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Research Article

Schur-Weyl Duality Theorem

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Abstract
This paper introduce the idea of Schur Weyl Duality Theorem which is a mathematical theorem in representation theory. And it has a numerous applications in quantum information theory. Specifically, in classical information theory, the method of types can be used to carry out some tasks such as estimating probability distribution, randomness concentration and data compression.

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1. Introduction

Schur-Weyl duality is a powerful tool from representation theory of group. It has been shown that Schur basis can be used to generalize classical method of types, thus allowing us to perform quantum counterparts of the previously mentioned tasks. More explicit results related to Schur-Weyl duality can be mentioned. For example, Keyl and Werner using Schur-Weyl duality to estimate the spectrum of an unknown ($d - level$) mixed state ρ from its $n - fold$ product state [1]. Harrow gave efficient quantum circuits for Schur and ClebschGordan transforms from computational point of view [2, 3]. Obviously, a Hilbert space of interest in quantum information is $(C^d)^{\otimes n}$. Thus, briefly the idea of Schur-Weyl duality theorem is that considering the tensor space $(C^d)^{\otimes n}$, this tensor space decomposes into a direct sum of tensor products of irreducible representations for the following groups S_n and $U(d)$.

2. Schur-Weyl Duality Theorem

In this section we will introduce the ancillary results that is for sure play a great role to really understand Schur-Weyl duality theorem, specifically these results are well known facts in representation theory as well as in finite group (compact group) for more details see [4]. Firstly in order to understand the following proposition we need to know that a von Newmann algebra, it is just a special kind of C^* algebra.

Proposition 1. *Supposed that V and W be finite dimiontional complex vector space, then if $A \subseteq End(V)$ and $B \subseteq End(W)$ are von Newmann*
algebras, then
$(A \otimes B)^0 = A^0 \otimes B^0$
Proof. [5]
We need to show that $(A \otimes B)^0 \subseteq A^0 \otimes B^0$ since it is clear that $A^0 \otimes B^0 \subseteq (A \otimes B)^0$ Choose an arbitrary $T \in (A \otimes B)^0$ by an operator Schimidt decomposition, we have
$T = \mathrm{X}cjMj \otimes Nj$
j
for all $c_j \geq 0$ and M_j be orthonormal basis in $End(C^m)$ where $m \in A$

, N_j be orthonormal basis in $End(C^n)$ where $n \in B$ Now let
$A \otimes 1_W, 1_V \otimes B \in A \otimes B$
then we have
$[T, A \otimes 1_W] = 0 = [T, 1_V \otimes B]$
That is
$$\sum_j c_j[M_j, A] \otimes N_j = 0$$
and
$$\sum_j c_j M_j \otimes [N_j, B] = 0$$
drop these term for which c_j are zero. Thus c_j is positive for all j in above equation and because M_j, N_j are linearly independent it follows that
$[M_j, A] = 0$
and
$[N_j, B] = 0$
This implies that that
$Mj \in A0$
and
$Nj \in B0$
Thus $T \in A^0 \otimes B^0$
so we have proved that $(A \otimes B)^0 \subseteq A^0 \otimes B^0$ since $T \in A^0 \otimes B^0$ then we obtain that
$(A \otimes B)^0 = A^0 \otimes B^0$

Proposition 2 (The Dual Theorem)**.** *Let V be representation of finite group G (then is completely reducible). with decomposition*
$$V \cong \bigoplus_{\alpha \in \hat{G}} n_\alpha V_\alpha$$
$$V \cong \bigoplus_{\alpha \in \hat{G}} V_\alpha \otimes C^{n_\alpha}$$
Where $\alpha \in \hat{G}$ *labels an irreducible representations* V_α *and* n_α *is the multiplicity of an irreducible representations* V_α*. And let A be the algebra generated by V and let* $B = A^0$*, then we get*
$$A \cong \bigoplus_{\alpha \in \hat{G}} End(V_\alpha) \otimes 1_{C^{n_\alpha}}$$
and
$$B \cong \bigoplus_{\alpha \in \hat{G}} 1_{V_\alpha} \otimes End(C^{n_\alpha})$$
Furthermore we have $B^0 = A^{00} = A$*. that is the double commutant of A is just itself.*
2.0.1. *Note.* Before we just prove this theorem we will note that let $\hat{G}$ be complete set of inequivalent irreducible representations of G. then for any irreducible representation V_α of V, there is a basis under which the action of $V(g)$, $\forall g \in G$ can be written as $V(g) \cong \bigoplus_{\alpha \in \hat{G}} V_\alpha(g) \otimes 1_{n_\alpha}$
Now we will prove this theorem.
Proof. Now, sice A is algebra generated by V, so
$$d_\alpha/|G| \sum_{g \in G} \overline{V_{\alpha,ij}(g)} V(g) \in A$$
Then by orthogonality of functions $V_{\alpha,ij}$ and decomposition of V into irreducible representation we have

$$d_\alpha/|G|\sum_{g\in G}\overline{V_{\alpha,ij}(g)}V(g)$$

$$= d_\alpha/|G|\sum_{g\in G}\overline{V_{\alpha,ij}(g)}(\bigoplus_{\beta\in\widehat{G}}V_\beta(g)\otimes 1_{C^{n_\beta}})$$

$$= \bigoplus_{\beta\in\widehat{G}}(d_\alpha/|G|\sum_{g\in G}\overline{V_{\alpha,ij}(g)}V_\beta(g))\otimes 1_{C^{n_\beta}}$$

$$= \bigoplus_{\beta\in\widehat{G}}(d_\alpha/|G|\sum_{g\in G}\overline{V_{\alpha,ij}(g)}\sum_{k,l}V_{\beta,kl}(g)E_{\beta,kl})\otimes 1_{C^{n_\beta}}$$

$$= \bigoplus_{\beta\in\widehat{G}}(d_\alpha/|G|\sum_{g\in G}\overline{V_{\alpha,ij}(g)}\sum_{k,l}V_{\beta,kl}(g))E_{\beta,kl}\otimes 1_{C^{n_\beta}}$$

$= E\alpha,ij \otimes 1C n\alpha$

So

$= E\alpha,ij \otimes 1C n\alpha \in A$

then,

$= End(V_\alpha) \otimes 1_C n_\alpha \subseteq A$

Then

$A \;= span\{V(g)|g \in G\}$

$\tilde{=}\ ^{\mathrm{M}}span\{V_\alpha(g) \otimes 1_C n_\alpha\}$

$\alpha \in G_{\mathrm{b}}$

$= {}^{\mathrm{M}}End(V_\alpha) \otimes 1_C n_\alpha$

$\alpha \in G_{\mathrm{b}}$

Then

$B \;= A^0 = ({}^{\mathrm{M}}End(V_\alpha) \otimes 1_C n_\alpha)^0$

$\alpha \in G_{\mathrm{b}}$

$= {}^{\mathrm{M}}(End(V_\alpha))^0 \otimes (1_C n_\alpha)^0$

$\alpha \in G_{\mathrm{b}}$

$= \mathrm{M}1V\alpha \otimes End(Cn\alpha)$

$\alpha \in G_{\mathrm{b}}$

Now we can see that

$\mathrm{M}\ \ 0$

$1_{V\alpha} \otimes End(C_{n\alpha}) \subseteq A = B$

$\alpha \in G_{\mathrm{b}}$

Now by considering a system of n qudits, each with standard local computational basis $\{|i\mathrm{i}, i = 1,...,d\}$.

The Schur-Weyl duality relates transforms on the system performed by local d dimension unitary operations to those performed by permutation of the qudits.

The symmetric group S_n is naturally represented in our system by

$p(\pi)|i\mathrm{i} := |i\pi(1)-1...i\pi(n)-1\mathrm{i}$

where $\pi \in S_n$ and $|i_1...i_n\mathrm{i}$ is shorthand for $|i_1\mathrm{i} \otimes ... \otimes |i_n\mathrm{i}$. Let $U(d)$ denotes to the group of $d\times d$ unitary operators. This group is naturally represented in our system by

$Q(U)|i_1...i_n\mathrm{i} := U|i_1\mathrm{i} \otimes ... \otimes U|i_n\mathrm{i}$

where $U \in U(d)$

Thus, we have the following significant result:

Theorem 1 (Schur)**.** *Let*

$A \;= span\{P(\pi) : \pi \in S_n\}$

and

$B \;= span\{Q(U) : U \in U(d)\}$ *Then,* $A^0 = B$ *and* $A = B^0$

Proof. [5]. Let $\{|1\mathrm{i},...,|d\mathrm{i}\}$ be standard basis for C^d. And for an order n tuple

$I = (i_1,...,i_d)$ with $i_1,....i_n \in [d]$, $[d] := \{1,...,d\}$ Define $|I| = n$ and

$|I\mathrm{i} := |i_{1,...,in}\mathrm{i}$, the tensors

$\{|I\mathrm{i} : I \in [d]^n\}$

Give us basis for $(C^d)^{\otimes n}$. The representation $P(\pi)$ acts on these basis as the following

$P(\pi)|I\mathrm{i} = |\pi.I\mathrm{i}$

where $\pi \in S_n$, and $I = (i_1,...,i_n)$ So we define it by

$\pi.(i1,...,in) = (i\pi(1)-1,...,i\pi(n)-1)$

Clearly, π changes the position (1) to (n) of the indices, not their values (1) to (d) and we have $(\sigma.\pi).I = \sigma.(\pi.I)$, where $\sigma,\pi \in S_n$. Now, let $T \in End(C^d)^{\otimes n}$, and let its action relative to basis $\{|I\rangle, I \in [d]^n\}$ per basis element be given by the matrix $[t_{IJ}]$

$T.|J\rangle = {}^X t_{IJ}|I\rangle$

$I\in[d]^n$

Setting

$TP\pi|J\rangle = P(\pi)T|J\rangle$

It follows that T commutes with elements of A^0 if and only if

$t\pi.I\pi.J = tIJ$

(1)

$\forall I,J;\pi \in S_n$ Let $< X,Y >: End(C^d) \times End(C^d)$ 7−→ C denote the non-degenerate bilinear form $Tr(XY)$. The restriction to A^0 of this form is non-degenrate, we can show that by using the following projection : X 7−→ X^* of $End(C^d)^{\otimes n}$ onto A^0 given by avarging over

S_n ,

$X^* = 1/n!\ {}^X P(\pi)XP(\pi)^{-1}$

$\pi\in S_n$

If $T \in A^0$, then

$< X^*,T >= 1/n!\ {}^X Tr(P(\pi)XP(\pi)^{-1}T)$

$\pi\in S_n$

$= 1/n!\ {}^X Tr(P(\pi)XTP(\pi)^{-1})$

$\pi\in S_n$

$= 1/n!.n!Tr(XT)$

$=< X,T >$

Hence, $< A^0,T >= 0$ implies that $< X,T >= 0$ for all $X \in End(C^d)^{\otimes n}$, and so $T = 0$. Thus, the trace form on A^0 is non-degenerate. Now, to show that $A^0 = B$, it suffices to show that if $T \in A^0$ is orthogonal to B, then $T = 0$.

If $g = [g_{i,j}] \in GL_d(C)$ then $Q(g)$ has matrix:

$gI,J = gi_1,j_1...gi_n,j_n$

relative to basis $\{|I\rangle : I \in [d]\}$ so supposed that

$< T,Q(g) >= \text{X}tI,Jgi1,j1...gin,jn = 0$

I,J

(2)

for all $g \in GL_d(C)$.

Now, we define polynomial function f_T on $M(C^d)$ by

$fT\,(X) = \text{X}tI.Jxi1,j1...xin,jn$

$I.J$

$\forall X = [x_{ij}] \in M(C^d)$. The polynomial function f_T must be identically zero because f_T is continues on $M(C^d)$, so.

$f_T(X) = 0$

that is

X

$tIJxi_1,j_1...xi_n,j_n = 0$

$I.J$

(3)

Then we will be shown that from equation (1)*and*(3), it follows that $t_{I,J} = 0$, $\forall I,J$ and thus $T = 0$ and to achieve this, equation (3) will be rewritte according to distinct monomials in matrix entries $\{x_{ij}\}$.

Let $X_{IJ} = x_{i1,j1}...x_{in,jn}$ and view this monomial as polynomial function on $M(C^d)$, and let Ξ be set of all order pairs (I,J) of multi-indices with $|I| = |J| = n$. Then S_n acts naturally on Ξ by

$\pi.(I,J) = (\pi I,\pi J)$

$\forall \pi \in S_n$ and this action defines an equivalence relation $\sim$ on Ξ where

$(I,J) \sim (I^0,J^0)$

if and only if

$(I^0,J^0) = (\pi I,\pi J)$

Choose a set Γ of representatives of equivalence classes. Let (I,J) be any pair of multi-indices in Ξ. Then there is a $\gamma \in \Gamma$ so that $(I,J) \in [\gamma]$. In fact, since variable $x_{i,j}$ matually commute we have

$x\gamma = x\pi.\gamma$

Now we will show that γ is uniquely determined by x_γ and firstly let supposed that

$xI,J = xI0,J0$

then there must be integer p such that

$xi_10,j_10 = xi_p,j_p$

Call $p = 1^0$, similarly, there must be integer q 6= p such that

$xi_20,j_20 = xi_q,j_q$

Call $q = 2^0$ and if we keep counting in this way we get permutation

$\pi : (1,2,...,n) \longmapsto (1',2',....,n')$

such that

$$I = \pi . I^0$$

and

$$J = \pi J^0$$

Now for $\gamma \in \Gamma$ let $n_\gamma = |S_n.\gamma|$ be cardinality of the corresponding orbit. Assuming that if the coffecients t_{IJ} satisfy equation (1) and (3) then as $t_{I,J} = t_\gamma$ for all $(I,J) =\in S_n.\gamma$ it follows that

$$\sum_{IJ} x_{i1,j1} \dots x_{in,jn} = \sum_{\gamma \in \Gamma} n_\gamma t_\gamma x_\gamma = 0$$

The linear indebendence of the set of monomials $\{x_\gamma | \gamma \in \Gamma\}$ then implies that for all $\gamma \in \Gamma$ we have $n_\gamma t_\gamma = 0$ But per definition $n_\gamma \geq 1$, so it means that $t_\gamma = 0$ and thus $t_{i,j} = 0$ for all $(I,J) \in \Xi$ so, we have proved that $T = 0$ then we get $B = A^0$.

Finally now we will introduce the main result in this project with a wonderful decomposition of the representations on n folds tensor space $(C^d)^{\otimes n}$ of $U(d)$ and S_n, respectively by using their irreducible representations.

Theorem 2 (Schur Duality Theorem)**.** *There is a basis, called a Schur basis, in which the representation* $(QP,(C^d)^{\otimes n})$ *of* $U(d) \times S_n$ *decompose into irreducible representations* Q_λ *and* P_λ *of* $U(d)$ *and* S_n*, respectively such that For every* $\lambda \vdash (n,d)$

$$(C^d)^{\otimes n} \cong \bigoplus_{\lambda \vdash (n,d)} Q_\lambda \otimes P_\lambda$$

$$P(\pi) \cong \bigoplus_{\lambda \vdash (n,d)} 1_{Q_\lambda} \otimes P_\lambda(\pi)$$

$$Q(U) \cong \bigoplus_{\lambda \vdash (n,d)} Q_\lambda(U) \otimes 1_{P_\lambda}$$

Clearly these representations are commutant with each other, such that

$$[Q(U),P(\pi)] = 0$$

so we can define representation $(QP,(C^d)^{\otimes n})$ *of* $U(d) \times S_n$ *as:*

$$QP(U.\pi) = Q(U)P(\pi) = P(\pi)Q(U)$$

for all $(U,\pi) \in U(d) \times S_n$ *then*

$$QP(U.\pi) = U^{\otimes n} P_\pi = P_\pi U^{\otimes n}$$

$$\cong \bigoplus_{\lambda \vdash (n,d)} Q_\lambda(U) \otimes P_\lambda(\pi)$$

2.0.2. *Note.* Obviously that algebras generated by P and Q centralized each other. So we apply double commutant theorem to get the last above equation.

Proof. By application of the dual and Schur theorem when $G = S_n$ and to its dual partner $U(d)$ we will get the three above equations, where P_λ is irreducible representations of S_n. To see that $Q(X) \in B$ where $X \in End(C^d)$ the decomposition of $Q(X)$ is of form

$$Q(X) \cong \bigoplus_{\lambda \vdash (n,d)} Q_\lambda(X) \otimes 1_{P_\lambda}$$

Thus

$$QP(X,\pi) = X^{\otimes n} P(\pi) = P(\pi) X^{\otimes n} \cong \bigoplus_{\lambda \vdash (n,d)} Q_\lambda(X) \otimes P_\lambda(\pi)$$

so now we just need to prove that Q_λ is irreducible representation too. And we will prove that by two ways, it will be rephrasing the one that given in [6].

First proof;

Q_λ is irreducible representation if and only if its extension to $GL_d(C)$ is irreducible, that is, $Q_\lambda(U(d))$ is irreducible representation if and only if $Q_\lambda(GL_d(C))$ is irreducible. Thus we will show that Q_λ is indecomposable under $GL_d(C)$.

Using Schur's theorem this is equivalent to show that

$$End_{GL_d(C)} \cong C$$

i.e maps in $End(Q_\lambda)$ that commutant with the action of $GL_d(C)$ are identity so, by Schur theorem :

$$End_{S_n}(C^d)^{\otimes n} \cong \bigoplus_\lambda End(Q_\lambda) \otimes 1_{P_\lambda}$$

$$\cong \bigoplus_\lambda End(Q_\lambda)$$

so

$$End_{GL_d(C) \times S_n}(C^d)^{\otimes n} \cong \bigoplus_\lambda End_{GL_d(C)}(Q_\lambda)$$

Now by dual theorem $GL_d(C)$ and S_n are double commutant

$$End_{S_n}(C^d)^{\otimes n} = span\{T^{\otimes n} | T \in GL_d(C)\}$$

Thus $End_{GL_d(C) \times S_n}(C^d)^{\otimes n}$ is contained in center of $End_{S_n}(C^d)^{\otimes n}$ So $End_{GL_d(C)}(Q_\lambda)$ is contained in center of $End_{Q_\lambda} \cong C$ then finally

$$End_{GL_d(C)}(Q_\lambda) \cong C$$

Second proof;

As we have representation of finite group S_n which is then compact group, $P(\pi)$ is equivalent to direct sum of irreducible representations such that
$P(\pi) \sim {}^{\mathrm{M}}1_{m\lambda} \otimes P_\lambda$
Scn
$A = span(P(\pi)) \;\tilde{=}\; {}^{\mathrm{M}}1_{m\lambda} \otimes span(P_\lambda)$
Scn
$= {}^{\mathrm{M}}1_{m\lambda} \otimes End(P_\lambda)$
Scn
where m_λ is mutually of P_λ in $P(\pi)$ so this give rise to the following decomposition
$(Cd)\otimes n \sim= \mathrm{M}Cm\lambda \otimes (P\lambda)$
Scn
Now since we are working in the "Schur-Weyl basis", we have actually the following equality
$Q(U)P(\pi) = {}^{\mathrm{M}}Q_\lambda(U) \otimes P_\lambda(\pi)$
λ
Let $T_\mu \in End_{U(d)}$, and we have already that
$Q(U) = {}^{\mathrm{M}}Q_\lambda(U) \otimes 1_{p\lambda}$
λ
Then
$(T_\mu \otimes 1_{p\mu}).Q(U) = Q(U).(T_\mu \otimes 1_{p\mu})$
To see that;
$(T_\mu \otimes 1_{p\mu}).Q(U) = (T_\mu \otimes 1_{p\mu})({}^{\mathrm{M}}Q_\lambda(U) \otimes 1_{p\lambda})$
λ
$= (T_\mu \otimes 1_{p\mu})(Q_\mu(U) \otimes 1_{p\mu})$
$= (Q_\mu(U) \otimes 1_{p\mu})(T_\mu \otimes 1_{p\mu})$
$= Q(U).(T_\mu \otimes 1_{p\mu})$
So,
$(T_\mu \otimes 1_{p\mu}) \in Q(U)^0 = B^0 = A$
However;
$A = {}^{\mathrm{M}}1_{Q\lambda} \otimes End(P_\lambda)$
λ
Then
$T_\mu \otimes 1_{P\mu} + 1_{Q\mu} \otimes End(P_\mu)$
So we obtained that
$T\mu \in C1Q\mu$ Thus we have proved that Q_μ is irreducible representation of $Q(U)$

3. example
Let $n = 2$ and d is greater than one. Then the Schur-Weyl duality is the statement that the space of two-tensors decomposes into symmetric and antisymmetric parts, each of which is also an irreducible representation for $U(d)$, when $n = 2$ we get the following
$Cd \otimes Cd = \mathrm{M}\; P\lambda \otimes Q\lambda$
$\lambda`(2,d)$
$Cd \otimes Cd \sim= (Ptrivial \otimes Utrivial) \oplus (Psign \otimes Usign)$
$C^d \otimes C^d \;\tilde{=}\; sign(C^d \otimes C^d) \oplus \Lambda_2 C^d$
where we define anti-symmetric subspace $\Lambda_2 C^d$ is the subspace of$(C^d)^{\otimes n}$ of all vectors that are negated by odd permutations :
$\Lambda_2 C^d = \{v \in (C^d)^{\otimes n} : P_n(\pi)v = sgn(\pi)v\}$
for all $\pi \in S_n$.

So, in other words The symmetric group S_2 consists of two elements and has two irreducible representations, the trivial representation and the sign representation. The trivial representation of S_2 gives rise to the symmetric tensors, which are invariant (i.e. do not change) under the permutation of the factors, and the sign representation corresponds to the skew-symmetric tensors, which is just flip the sign.

4. Conclusion

It has been shown that there is a correspondence between the $P(\pi)$ representation and $Q(U)$ representation described in the statement of the SchurWeyl Duality Theorem. It is achieved through the study of the irreducible representations of P_λ and Q_λ a thorough treatment of this theory and possible applications of SchurWeyl Duality can be found in [2].

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